

# UNIFORMITY IN THE MORDELL-LANG CONJECTURE

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## INTRODUCTION

The Mordell-Lang conjecture and its variants assert subvarieties of algebraic groups can meet certain subgroups only in a finite set of translates of subgroups. In most cases the number of such translates is unknown and it is not even known whether this number may be bounded by a function of the geometric data. In this note we show that some uniformity follows immediately from the finiteness result.

The main technical result behind this note is a theorem of Pillay on the stability of the theory of an algebraically closed field with a predicate for a group of Lang type [4]. My interest in these questions was renewed through my reading of the paper [5] in which a version of uniformity for the Mordell-Lang conjecture for abelian varieties is proved. As the reader will see the main result of this note is an immediate corollary of Pillay's theorem, but somehow it was not noticed earlier.

## MAIN THEOREM

We begin by stating more precisely what we mean by an analogue of the Mordell-Lang Conjecture.

**Definition:** We say that the subgroup  $\Gamma$  of an algebraic group  $G$  over an algebraically closed field  $K$  is of *Lang type* if

- for any subvariety  $X$  of a Cartesian power  $G^n$  of  $G$  the set  $X(K) \cap \Gamma^n$  is a finite union of cosets of subgroups.

We say that  $\Gamma$  is *uniformly of Lang type* if in addition

- for any algebraic family  $\{X_b\}_{b \in B}$  of subvarieties of  $G^n$  there is a sequence  $H_1, \dots, H_m$  of algebraic subgroups of  $G$  such that for any  $b \in B$  there are a set  $I \subseteq \{1, \dots, m\}$  and points  $\gamma_1, \dots, \gamma_m \in \Gamma^n$  such that the Zariski closure of  $X_b(K) \cap \Gamma^n$  is  $\bigcup_{i \in I} \gamma_i + H_i$ .

There are a number of known groups of Lang type.

### Example:

- If  $G$  is a semiabelian variety defined over a field  $K$  of characteristic zero and  $\Gamma \leq G(K)$  is a subgroup of finite dimension ( $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} < \infty$ ), then  $\Gamma$  is of Lang type [1, 7, 3].
- If  $A$  is an abelian variety over a field  $K$  of characteristic  $p$  admitting no non-trivial homomorphisms of algebraic groups to abelian varieties defined over a

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finite field and  $\Gamma \leq A(K)$  is a subgroup of finite  $\mathbb{Z}_p$ -rank ( $\text{rk}_{\mathbb{Z}_p} \Gamma \otimes \mathbb{Z}_p < \infty$ ), then  $\Gamma$  is of Lang type [2].

- If  $K$  is a field of positive characteristic and  $\Gamma \leq K$  is torsion module of a Drinfeld module of generic characteristic, then  $\Gamma$  is of Lang type [6].

Recall that a group  $G$  considered as an  $\mathcal{L}$ -structure for some first-order language  $\mathcal{L}$  is said to be *modular* if for every natural number  $n$  every  $\mathcal{L}$ -definable subset of  $G^n$  is a finite Boolean combination of cosets definable subgroups.

Recall that if  $\mathfrak{M}$  is an  $\mathcal{L}$ -structure for some first-order language  $\mathcal{L}$  and  $X \subseteq M^n$  is a nonempty subset of some Cartesian power of  $M$ , then the full-induced structure on  $X$  is the structure  $\mathfrak{X}$  having universe  $X$  and basic relations  $D \cap X^m$  for each  $\mathcal{L}_M$ -definable  $D \subseteq M^{nm}$ .

We could restate the definition of  $\Gamma$  being of Lang type as *the full induced structure on  $\Gamma$  from  $(K, +, \cdot)$  is modular*.

**Theorem 1.** *If  $\Gamma$  is of Lang type, then it is uniformly of Lang type.*

*Proof.* By Proposition 2.6 of [4], the structure  $(K, +, \cdot, \Gamma)$  is stable and the formula  $x \in \Gamma$  is one-based. Suppose that  $\Gamma$  is not uniformly of Lang type witnessed by some family  $\{X_b\}_{b \in B}$  of subvarieties of  $G^n$ . Then for any finite sequence  $B_1, \dots, B_m$  of subgroups of  $\Gamma^n$  there is some  $b \in B$  for which the formula  $(\forall x_1, \dots, x_m \in G^n) \bigwedge_{I \subseteq \{1, \dots, m\}} (\exists y \in \Gamma^n) [(y \in X \wedge \bigwedge_{i \in I} y \notin x_i + B_i) \vee (y \notin X \wedge \bigvee_{i \in I} y \in x_i + B_i)]$  holds. By compactness, we can find some elementary extensions  $(^*K, +, \cdot, ^*\Gamma)$  and a point  $b \in B(^*K)$  satisfying all of these formulas for all choices of finite sequences of subgroups of  $\Gamma^n$ . As  $\Gamma$  is one-based as a definable group in  $(K, +, \cdot, \Gamma)$ ,  $^*\Gamma$  is also one-based. Thus, every (parametrically) definable (in  $(^*K, +, \cdot, ^*\Gamma)$ ) subset of  $^*\Gamma^n$  is a finite Boolean combination of cosets of  $(K, +, \cdot, \Gamma)$ -definable subgroups. Thus,  $X_b(^*K) \cap ^*\Gamma^n$ , being a definable subset of  $^*\Gamma^n$ , is a finite Boolean combination of cosets of  $(K, +, \cdot, \Gamma)$ -definable subgroups of  $^*\Gamma^n$ . As  $X_b$  is Zariski closed, this combination is necessarily a finite union. Thus, there are subgroups  $B_1, \dots, B_m$  of  $\Gamma$  and points  $a_1, \dots, a_m \in G(^*K)$  such that  $(^*K, +, \cdot, ^*\Gamma) \models (\forall y \in ^*\Gamma^n) (y \in X_b \leftrightarrow \bigvee_{i=1}^m y \in a_i + B_i)$ . This contradicts the choice of  $b$ .

Therefore,  $\Gamma$  is uniformly of Lang type.  $\square$

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